Actions of fusion categories on topological spaces

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January 26, 2021

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- Can always be realized as representation category of a **finite quantum groupoid** (e.g. C* weak Hopf algebra).
- Fusion rules: $[X \otimes Y] = \bigoplus_{[Z]\in \mathrm{Irr}(\mathcal{C})} N_{XY}^{Z}[Z]$
- Fusion categories have unique positive dimension function on objects which is a character of the fusion ring, i.e. $d_{\mathbb{1}} = 1$ and $d_X d_Y = \sum_Z N_{XY}^{Z} d_Z$ (need not be integer!)

Correspondences

IF A is a C*-algebra, and $A - A$ correspondence is a (right) Hilbert A-module X_A together with a non-degenerate homomorphism $A \to L(X_A)$ (adjointabe operators).

Correspondences assemble into a C*-tensor category **Corr**(A):

- Objects are correspondences X_A
- Morphisms are (adjointable) intertwiners.
- The $\otimes$ product is given by the relative product of correspondences $X \boxtimes_A Y$.

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- Subfactors constructions work to produce actions of fusion catgeories on C*-algebras realted to **graphs**: AF algebras, Graph C*-algebras, Free graph algebras, etc.

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- Every fusion category can act on finite discrete set via finitely semi-simple module categories (and thus on disconected spaces).
- It is not obvious whether or not we can build actions of fusion categories (with no fiber functor) on $C(X)$ where X is connected...

Obstructions

Obstruction Theorem

Let A be a unital, stably finite C*-algebra, and $\mathcal{C}$ a fusion category acting on A. Then there exists a state ϕ on the ordered abelian group $K_0(A)$ such that $\phi(K_0(A)) \subseteq \mathbb{R}$ contains the dimensions of objects in $\mathcal{C}$.

Idea of proof:

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- Satisfies $\phi([H] \triangleleft [X]) = d_X \phi([H])$. for all $[H] \in K_0^+(A)$.

Corollary

If X is a compact connected Hausdorff space and A is a unital continuous trace C*-algebra with spectrum A (e.g. $C(X)$) then any fusion category acting on A must have integral dimensions.

- Can we find actions of integral fusion categories **with no fiber functor** on connected spaces?
- What are examples of such fusion categories? (Can't come directly from quantum groups!)

3-cocycles

Let G be a (discrete) group. A *unitary 3-cocycle* is a function

$$\omega : G \times G \times G \to \mathrm{U}(1)$$

such that

$$\omega(f,g,h)\omega(f,gh,k)\omega(g,h,k) = \omega(fg,h,k)\omega(f,g,hk)$$

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- ω is a *coboundary* if there exists $c : G \times G \to \mathrm{U}(1)$ such that $\omega(f,g,h) = c(g,h)^{-1}c(fg,h)c(f,gh)^{-1}c(f,g)$. Such a c is a called a *trivialization* of ω.

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- The set of 3-cocycles module the set of coboundaries is the abelian group $H^3(G,\mathrm{U}(1))$

Hilb(G, ω)

Associated to a 3-cocycle $\omega \in Z^3(G, \mathrm{U}(1))$ is a fusion category Hilb(G, ω):

- Objects are G-graded (finite dimensional) Hilbert spaces, morphisms are linear maps respecting grading.
- $V_g \otimes W_h := (V \otimes W)_{gh}$
- Asscoiator isomorphism $(V_g \otimes W_k) \otimes U_k = ((V \otimes W) \otimes U)_{ghk} \cong (V \otimes (W \otimes U))_{ghK} = V_g \otimes (W_h \otimes U_k)$ is given by scalar $\omega(g, h, k)$ times the usual associator in Hilb.
- Representations of Quasi-hopf algebra (Fun$(G), \omega)$.
- These have fiber functors (act on a point) if and only if $[\omega]$ is trival in $H^3(G, \mathrm{U}(1))$.

Anomalous symmetries

Definition: anomalous action

Let G be a group, $\omega \in Z^3(G, \mathrm{U}(1))$. An ω-anomalous action of G on a C*-algebra B consists of an assignment $g \mapsto \alpha_g \in \mathrm{Aut}(B)$, together with a family of unitaries $m_{g,h} \in M(B)$ satisfying:

- $m_{g,h}\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)m_{g,h}$ for all $x \in B$
- $\omega(g, h, k)m_{gh,k}m_{g,h} = m_{g,hk}\alpha_g(m_{h,k})$

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- For G finite, these are the same thing as actions of the fusion category $\mathrm{Hilb}(G,\omega)$ on B such that each $g \in G$ is assigned "automorphic bimodule".
- From an arbitrary $\mathrm{Hilb}(G,\omega)$ action on A, we can pass to an ω-anomalous action of G on the stabilization $A \otimes \mathcal{K}$ (latter is compact operators on separable Hilbert space).

Examples

- For $Z(MB) = \mathbb{C}1$, and ω-anomalous action is the same thing as a homomorphism $\pi : G \to \mathrm{Out}(B)$ whose "lifting obstruction" is precisely ω (we are interested in non-trivial centers!)
- (V. Jones): For every finite group G and every anomaloy ω, there exists a (essentially unique) ω-anomalous G-action on the hyperfinite II_1 factor.
- Examples of actions on C*-algebras from general fusion category constructions (e.g. on AF-algebras, Cuntz-Kreiger algebras, free graph algebras etc.)
- Consider G as a discrete set. Then for any group G and any $\omega \in Z^3(G, \mathrm{U}(1))$, there exists an ω-anomlous action on $c_0(G)$. Purely algebraic.

Anomalies in topology

No-go Theorem

Let X be a compact connected, locally path-connected Hausdorff space.

- If $H^1(X, \mathbb{Z}) = 0$ then there are no anomalous actions on $C(X)$ for any finite group G.
- If in addition X has no non-trivial complex line bundles (for manifolds $H^2(X, \mathbb{Z}) = 0$, e.g. homology spheres of dimension $n \geq 2$), there are no anomalous actions of any finite groups on the stabilization $C(X) \otimes \mathcal{K}$.

Go Theorem

For every finite group G, every $\omega \in Z^3(G, \mathrm{U}(1))$, and every $n \geq 2$, there exists a closed connected n-manifold M and an ω-anomalous action of G on $C(M) \otimes \mathcal{K}$. For $n \geq 4$, M can be chosen so that $H^1(M, \mathbb{Z}) = 0$

Anomalies in coarse geometry (mention in passing!)

No-go Theorem

Let X be a discrete metric space with bounded geometry. Then there are no anomalous actions of any group on the Roe algebra $C^*(X)$.

Go Theorem

For every finite group G and $\omega \in Z^3(G, \mathrm{U}(1))$, there exists a discrete metric space X with bounded geometry and property A, and an ω-anomalous action of G on the Roe corona $C^*(X)/\mathcal{K}$.

Idea of construction

Theorem (Adaptation of Eilenberg-MacLane (groups), V. Jones (von Neumann algebras) to C*-setting)

Suppose we have the following data:

- A group Q and $[\omega] \in H^3(Q, \mathrm{U}(1))$, with a normalized representative $\omega \in Z^3(Q, \mathrm{U}(1))$.
- A group G and a surjective homomorphism $\rho : G \to Q$ with kernel K
- A normalized cochain $c \in C^2(G, \mathrm{U}(1))$ such that $dc = \rho^*(\omega)$.
- A homomorphism $\pi : G \to \mathrm{Aut}(B)$.

Then there exists an ω-anomalous action of Q on the twisted (reduced) crossed product $B \rtimes_{\pi,c} K$, where $c \in Z^2(K, \mathrm{U}(1))$ is the restriction of c to K.

How to use it?

Cohomology lemma (extension of V. Jones' lemma)

Let Q be a finite group and $[\omega_0] \in H^3(Q, \mathrm{U}(1))$. Then there exists a finite group G, a surjective homomorphism $\rho : G \to Q$, a normalized unitary 2-cochain $c \in C^2(G, \mathrm{U}(1))$, and a normalized unitary 3-cocycle representative $\omega \in [\omega_0]$ such that $dc = \rho^*(\omega)$ and $c|_{Ker(\rho)} = 1$.

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To find ω-anomalous action of Q on $C(M) \otimes \mathcal{K}$:

- Find free G-action on compact connected manifold $\tilde{M}$ (standard algebraic topology).

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- Use above theorem/lemma to obtain anomalous Q action on (ordinary, untwisted) crossed product $C(\tilde{M}) \rtimes \mathrm{Ker}(\rho)$ (if crossed product twisted, possible non-trivial Dixmier-Douady class).
- By the results of Green, $(C(\tilde{M}) \rtimes \mathrm{Ker}(\rho)) \otimes \mathcal{K} \cong C(\tilde{M}/\mathrm{Ker}(\rho)) \otimes \mathcal{K})$.

Continuous trace

- (unital) Continuous trace C*-algebras are (roughly) bundles of matrix algebras over compact Hausdorff spaces (which is its spectrum).
- Classified up to Morita equivalence by (torsion) elements of $H^3(X, \mathbb{Z})$.
- Only integral fusion categories can act on continuous trace C*-algebras with connected spectrum.
- Using Q-system completion (joint w/ Quan Chen, Roberto Hernandez Palomares, Dave Penneys) we can show, for every group theoretical fusion category $\mathcal{C}$ (Morita equivalent to Hilb(G, ω)) and every $n \geq 2$, there is a closed connected n-manifold X and an action on a continuous trace C*-algebras A with spectrum X and an action of $\mathcal{C}$ on A.